**TASK 1**

A line with slope 5 passes through the vertex of this parabola. Does it intersect the parabola in another point (other than the vertex)? If so, find the point of intersection. If not, explain why.



1. Write complete solutions that can be shown on the document camera and that show conceptual understanding of the problem. Provide multiple solutions from your Team.
2. Describe the mathematical ideas and procedures used in your solutions and how these contributed to your solution path.

* Graphical solution: We continued each graph on another coordinate plane and found the solution, then checked it in the algebraic equations of the next solution. This uses the equivalence of the algebraic and graphical representations of the functions. The intersections of the graphs are the graphical representations of when the algebraic solutions are the same. That is, the points (-2, -1) and (3, 24) are on both graphs, and also satisfy both equations. We didn’t look for another solution since the parabola is growing much faster than the line, so it won’t intersect again.
* Algebraic solution 1: We wrote the equation of parabola in vertex form and equation of line in point-slope form, isolated y, then set the two expressions equal to each other to solve. The initial ideas are that we found algebraic equations that represent the relationships between the x and y values of all points on the graph, and when we set the two expressions for y equal to each other, we created an equation that constrains the relationship to when the two equations have the same output value, y. Then, we noticed that the structure was nearly the same on both sides, that we could add 1 to both sides, then divide by (x+2), keeping in mind that we already have the solution when x=-2, and found x=3. Substituting that back into the line, we got y=24, so (3, 24) is the other intersection.
* Algebraic solution 2: We wrote the equation of the parabola in factored form, seeing that the quadratic function must be zero when x=-3 and x=-1, and also noticing that the function increased by 1 when the x-value increased by 1, so that the leading coefficient of the quadratic function must be 1. We used the linear function from our first algebraic solution, but wrote it in slope-intercept form, then, like above set the expressions for y equal to each other. We noticed that we had a quadratic equation with no noticeable way to make it efficient to solve, so expanded the left side and subtracted the right side to get zero alone. Note that the idea here is that when P(x)=Q(x), then P(x)-Q(x)=0. That is, the quadratic equation that results from subtracting the expression for the line from the expression for the parabola, results in an expression for a parabola that has roots that have the same x-values as the intersections of the original parabola and line. We can also think of the resulting quadratic equation as the intersection of the new parabola and the line y=0. We solved by factoring and found the same solution as in the first two solutions.
* Algebraic solution 3: We shifted the parabola to have vertex at the origin and noticed that the equation is y=x2, then looked at when the line y=5x would intersect it. This results in a simple equation which we divided by x (knowing that we didn’t care any more about when x=0) and found x=5. So the solution of our equation is (5, 25), but when we shift it back to the original graph, we have to move left two and down 1, giving us the intersection (3, 24).
* Table solution: We had our equations from above, but rather than set them equal, we created a table that showed how y was increasing as x increased in increments of 1 starting from the first intersection point (-2, -1). We found it at (3,24) using the idea that when we input the same x-value into each equation, and the same output value appeared in both functions, then the graphs of the two functions both passed through the point. Note that we feel lucky that the intersection had an integer x-and y-values, but we were aware that the line was above the parabola when x= -1, since the y-value of the line was greater than the y-value of the parabola, and were paying attention to the relative values of each as we continued the table, so that if the two y-values did not become equal, but the y-value of the parabola became greater than the y-value of the line, we could look between the two x-values for the intersection. This uses the idea that both functions are continuous.

1. Which of the ideas and procedures in (2) are the most important in order to answer the question with conceptual understanding?

* The equations of the functions represent the same set of points as the graphs.
* The equation created by setting the expressions for y equal to each other is the same as finding the point of intersection graphically and asking which x-value this occurs at.
* Interpreting the solution of the resulting equation as the x-value of the intersection, and finding the corresponding y-value, and knowing that this represents the intersection of the graphs. This idea is also in the table.

1. How do the other mathematical ideas identified in (2) support these important ideas?

* Being able to create equations that represent the graphs of the two initial functions is necessary to be able to set them equal to each other to find their intersections algebraically.
* Interpreting the solution of the resulting quadratic equation as the intersection relies on understanding the reason for setting them equal.

**Intersections**

**TASK 2**

Think of all possible lines that pass through the vertex of the parabola shown below. Which lines intersect the parabola again at another point and which ones do not? Explain.



1. Write complete solutions that can be shown on the document camera and that show conceptual understanding of the problem. Provide multiple solutions from your Team.
2. Describe the mathematical ideas and procedures used in your solutions and how these contributed to your solution path.

* Graphical/intuitive solution: We used a pencil to represent a line and after holding it horizontally through the point (-2, -1), we realized it could not intersect the parabola again, since the vertex is a minimum at that point.

Then, we started twirling the pencil (holding it at (-2, -1)) in a counter-clockwise direction to determine when the line intersects the parabola at another point. It is obvious the line with slope 1 intersects the parabola again at (1, 1), but to convince ourselves that it intersects more than once for all slopes between 0 and 1, we found the equation of the parabola and used a graphing calculator and small slopes for m in the equation y=m(x+2)-1. Then, we argued that if there was another line through (-2,-1) that is tangent to the graph of the parabola and that has a different slope than 0, then there are two distinct tangents to the graph at that point, which would indicate that the parabola had two distinct rates of change at that point, which we decided could not happen.

As we continued twirling our pencil counterclockwise, it was easy to see that slopes of 1 and 2 (and all slopes between 1 and 2) would result in lines that would intersect the parabola again. We believe the line, until it becomes vertical, will always intersect the parabola again, since the rate of change of the parabola is increasing but the rate of change of a line is constant. When the pencil became vertical we knew the line and parabola would not intersect again because the parabola is a function and could not have two different y-values for x=-2. Then, when the pencil continued counterclockwise, the slopes were negative, and because of the symmetry of the parabola over x=-2, the arguments are the same. Thus, the line and parabola will intersect twice for every slope of the line except when the slope is 0 or when the slope is undefined.

* Algebraic solution 1: We found an equation for the parabola in vertex form and a line with slope m that passes through (-2, -1) and set them equal. This uses the idea that when the two expressions for y are equal, the graphs will intersect at the resulting x-value (x changes from being a *variable* representing all real numbers to an *unknown* representing the value of x when the y-values of the equations are the same). We solved this equation to find a relationship between the slope and the x-value for the intersection: the x-value of the intersection is always 2 less than the slope of the line. To find the corresponding y-value, we let x=m-2 in the equation for the parabola and found that when the slope of the line is m, the line and the parabola will intersect again at (m-2, m2-1). Since this ordered pair makes sense for all real values of m, the second intersection point exists for all real values of x (except when m=0 since we eliminated that possibility when solving the equation).
* Algebraic solution 2: Same as algebraic solution 1, but subtracted to get zero on one side, then factored to solve.
* Algebraic solution 3: Same as algebraic solution 1, except we did not divide by x+2 because we wanted to see what happened if we solved the equation without eliminating the solution (-2, -1). Then we solved it using the quadratic formula. This required being able to simplify the expression and interpret the result with m still in it rather than just numbers, a higher level of abstraction than just the numerical solutions.
* Table solution: Since this only uses integer values for the slope, it cannot show all possible slopes, but it gave us an idea for about how the slope and the intersections were related.

1. Which of the ideas and procedures in (2) are the most important in order to answer the question with conceptual understanding?

* Understanding that the rate of change of the parabola continues increasing for all values of x but the rate of change of any (non-vertical) line is constant.
* Knowing to create equations for the parabola and a line with slope m and set them equal to find the intersection.
* Interpreting the solution to the equation for the intersection in terms of the x-value of the intersection and the slope, m.
* Being able to use the quadratic formula.

1. How do the other mathematical ideas identified in (2) support these important ideas?

**Intersections**

**TASK 3**

Think of all possible lines with a slope of 5. Which of these lines intersect the parabola shown below? How many times? Explain.



1. Write complete solutions that can be shown on the document camera and that show conceptual understanding of the problem. Provide multiple solutions from your Team.
2. Describe the mathematical ideas and procedures used in your solutions and how these contributed to your solution path.

* Graphical/intuitive solution: We sketched a line with slope 5 through the origin and saw that it could never intersect the parabola because it passed below and to the right of the parabola and as the graph of the parabola continues, it gets steeper and steeper so the line could never intersect it. We then imagined slowly moving the line up the y-axis as we kept the slope fixed at 5, and believe we will intersect the parabola exactly once somewhere before the y-intercept is 3, then, for all lines with slope 5 and y-intercept above that point, the line will intersect the parabola exactly twice.
* Algebraic solution 1: The equations of all lines with slope 5 are of the form y=5x+b, and we wrote the equation of the parabola, then set them equal to find the intersection point(s). It resulted in a quadratic equation with a constant of 3-b. There are 0, 1, or 2 real solutions to this equation where the number of solutions depend on the value of b (which is in the discriminant). When the values of b result in a negative discriminant, the line does not intersect the parabola; when the value of b results in exactly one solution, the line intersects the parabola in exactly one point (so will be tangent), and when the values of b result in a positive discriminant, there will be two intersections between the line and the parabola. A graphical connection is that the family of functions defined by y=P(x)-L(x), where P(x) defines the initial parabola, and L(x) defines the line with slope 5 and y-intercept b, is a set of congruent parabolas that are only vertically translated from each other. Therefore, the parabolas in this family that do not intersect the x-axis are the ones that correspond to 0 solutions in the quadratic equation, the parabola in this family with only one zero (whose vertex is on the x-axis) corresponds to the 1 solution in the quadratic equation, and the ones that intersect the x-axis twice correspond to two solutions in the quadratic equation.
* Algebraic solution 2: Like algebraic solution 1, but we completed the square, then solved for both b and x, interpreting it differently each time.
  + When we solved it for x, the expression is not a real number for b-values less than 2.75, so there are no lines with y-intercept below 2.75 that intersect the parabola. For b=2.75, we get exactly one x-value, ½, that corresponds to the x-value of the intersection of the line and parabola when the y-intercept of the line is 2.75. The line y=5x+2.75 is tangent to the parabola at the point (0.5, 5.25). For all values for b above 2.75, we get two solutions that define the x-values of the two intersection points.
  + When we solved it for b, we examined the structure of the resulting expression and found that b could only be at or above 2.75 (since the part of the expression that is squared will be a minimum of 0, and 2.75 is added to that).

1. Which of the ideas and procedures in (2) are the most important in order to answer the question with conceptual understanding?
2. How do the other mathematical ideas identified in (2) support these important ideas?